Journal of Statistical Physics, Vol. 32, No. 1, 1983

Comments on the Grad Procedure for the Fokker–Planck Equation*

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Received September 16, 1982

If the kinetic equation of a macroscopic system is expanded with respect to the velocity in terms of orthogonal functions, e.g., in terms of Hermite functions, one obtains an infinite hierarchy of equations for the expansion coefficients. Grad's method consists in truncating this hierarchy and investigating the remaining finite system. In this paper we set up conditions under which this procedure is rigorously justified in case of the Fokker–Planck equation.

KEY WORDS: Fokker–Planck equation of Kramers type; its hierarchy of Hermite equations of transfer; existence theory; truncation of the hierarchy.

1. INTRODUCTION

Series expansions of the (unknown) solutions of certain equations belong to the classical and widely used tools in theoretical and mathematical physics. By this method the original equation is transformed into a sequence of equations for the expansion coefficients. Among others, expansions in terms of orthogonal functions play a prominent role. It is sometimes useful to expand a function only with respect to a certain subset of its variables.

In applying this method one meets three characteristic problems:

1. One has to set up the equations for the expansion coefficients.
2. One has to solve these equations.
3. One has to show that the sum of the series of which the coefficients are gained by step 2 is a solution of the original equation.

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A special example of the method described is known as the Grad procedure. The basic equation is the kinetic equation of a macroscopic system. Its solutions are expanded with respect to the velocity in terms of Hermite functions. The coefficients depending on the space coordinate and the time obey the so-called Hermite equations of transfer. Originally Grad's method was introduced to get "approximate solutions" (whatever this means) for the Boltzmann equation. In this paper we apply the Grad expansion to the classical solution of a Fokker–Planck equation [see (1) below]. It is our aim to treat the first and the second problem stated above. To the third problem we will say only a few words.

We will arrive at the following results:

(1) In Section 2 by a rather simple calculation the Hermite equations of transfer are derived. Thus the first problem is solved.

(2) In Section 3 we study the solutions of the infinite hierarchy of equations of transfer. Conditions are given which ensure that the hierarchy can be solved recursively. This result generalizes for one-dimensional systems the existence theorem given in Ref. 1. Moreover the explicit form of the solution is given by a polynomial of operators acting upon certain functions. These results constitute a solution of the second problem.

(3) From the literature a lot of truncation procedures of infinite hierarchies are known (Cf., e.g., Refs. 2, 3, and the literature cited there). It is tacitly assumed that by these truncation procedures approximate solutions of the complete hierarchy of equations of transfer are gained. But up to now nothing has been proved. We will show that under rather weak conditions the solutions of the truncated system of equations are the first components of a solution of the complete system. This result justifies the usual method. But on the other hand it also shows that "truncation" has nothing to do with "approximation."

We remark that a similar result holds for the Boltzmann equation at least for Maxwellian molecules.

2. THE BASIC KINETIC EQUATION AND ITS HERMITE EQUATIONS OF TRANSFER

In this section we want to study the following (dimensionless) Fokker–Planck equation for a molecular distribution function f depending on (u, x, t):

$$\partial_t f + \nabla_x \cdot uf + \nabla_u \cdot Kf = \nabla_u \cdot \eta \cdot (\nabla_u + u) f \tag{1}$$

where (u, x, t) are dimensionless velocity, position, and time, respectively, where the dependence of the field of external forces on (u, x, t) is given by

$$K(u, x, t) = K_1(x, t) + K_2(x, t) \cdot u$$

and where the friction tensor η may depend on x, t. It is assumed that the system described by (1) is confined to a (not necessarily finite) box $\tilde{\Omega}_t$ which may vary in time. If I means the time interval for which the system is considered we have $(x,t) \in \bigcup_{t \in I} \tilde{\Omega}_t \times \{t\} =: \Omega$.

Now we will treat the first problem. Though we want to determine the coefficients of a series expansion of f in terms of Hermite functions we need not make use of the series expansion of f. The reason is that the Fourier coefficients of a function may exist even if this function is not expandable in an orthogonal series. This fact simplifies the problem very much. The only thing we need is the following.

Condition X. A function $f: \mathbb{R}^3 \times \Omega \to \mathbb{R}$ is said to satisfy Condition X if the following relations hold.

X.1. The integrals

$$m^n(x,t) := \int \overset{n}{\otimes} u f(u,x,t)\, du$$

exist for all $n \in \mathbb{N}_0$. The integral runs over *all of* $\mathbb{R}^3$.

X.2. In the interior Ω^0 of Ω and for $n \in \mathbb{N}_0$ the following equations are satisfied:

$$\int \overset{n}{\otimes} u \partial_t f\, du = \partial_t \int \overset{n}{\otimes} u f\, du \tag{2.1}$$

$$\int \overset{n}{\otimes} u u \cdot \nabla_x f\, du = \nabla_x \cdot \int \overset{n+1}{\otimes} u f\, du \tag{2.2}$$

$$\int \overset{n}{\otimes} u \nabla_u \cdot (Kf)\, du \qquad \text{exists} \tag{2.3}$$

$$\int \nabla_u \cdot \left(K \overset{n}{\otimes} u f\right) du = 0 \tag{2.4}$$

X.3. In the interior Ω^0 of Ω and for $n \in \mathbb{N}_0$ the Fokker-Planck collision operator satisfies the condition

$$\int \overset{n}{\otimes} u \otimes \nabla_u (\nabla_u + u) f\, du = \int \left[(\nabla_u^2 - u \otimes \nabla_u) \overset{n}{\otimes} u \right] f\, du$$

For the definition of ∇_u^2 see (39); it is not the Laplacian! For physical reasons it is quite natural to assume that a classical solution of (1) should satisfy Conditions X. For a more detailed discussion cf. Ref. 1, Section 2.1. The fields m^n are called moments of f. Since the tensors $\otimes^n u$ and the Hermite functions $\phi^n(u)$ are bijectively related, the moments m^n can be replaced by the Hermite moments h^n which are defined by

$$h^n(x,t) := \int \phi^n(u) f(u,x,t) \phi^0(u)^{-1}\, du =: \langle \phi^n, f(\cdot, x, t)\rangle_0 \tag{3}$$

Now using the notation introduced in Sections A.1–A.3 of the Appendix the first problem is solved by the following lemma.

Lemma. Let f be a classical solution of (1) satisfying Condition X. Then all h^n defined by (3) exist and satisfy the following equations:

$$\begin{aligned} \partial_t h^n + \nabla_x \cdot h^{n+1} + n\nabla_x \vee h^{n-1} - nK_1 \vee h^{n-1} - n\Pi_n K_2 \cdot h^n \\ - n(n-1)K_2 \vee h^{n-2} = -n\Pi_n \eta^T \cdot h^n, \qquad n \in \mathbb{N}_0, \quad h^{-1} = 0, \quad h^{-2} = 0 \end{aligned} \tag{4}$$

Proof. Let us abbreviate the left-hand side of (1) by $\mathscr{D}f$ and the right-hand side by $\mathscr{S}f$. Then for all $n \in \mathbb{N}_0$ both sides of the expression

$$\langle \phi^n, \mathscr{D}f\rangle_0 = \langle \phi^n, \mathscr{S}f\rangle_0 \tag{5}$$

are meaningful.

From Condition X.3 we conclude that

$$\langle \phi^n, \mathscr{S}f\rangle_0 = \langle \mathscr{S}\phi^n, f\rangle_0$$

With the help of $\mathscr{S}\phi^n = -n\Pi_n \eta^T \cdot \phi_n$ [cf. (48)] the last term of (4) follows at once.

From Condition X.2.1 the first term of (4) is read off immediately.

The term $\langle \phi^n, \nabla_x \cdot uf\rangle_0$ yields the second and the third term of (4) by use of Condition X 2.2. and formula (47). Finally we have to consider the term $\langle \phi^n, \nabla_u \cdot Kf\rangle_0$. Condition X.2.4 allows partial integration so that we find

$$\langle \phi^n, \nabla_u \cdot Kf\rangle_0 = A_1 + A_2$$

with

$$A_1 = -\langle K_1 \cdot \nabla_u H^n, f\phi^0\rangle_0$$

and

$$A_2 = -\langle u \cdot K_2^T \cdot \nabla_u H^n, f\phi^0\rangle_0$$

where $H^n = \phi^n/\phi_0$ denotes the nth Hermite polynomial [cf. (49)]. Using formula (50c), A_1 is seen to give the fourth term of (4). Using again formula (50c) and formula (50a) one finds

$$u \cdot K_2^T \cdot \nabla_u H^n = n(u \cdot K_2^T) \vee H^{n-1} = n\Pi_n K_2 \cdot H^n + n(n-1)K_2 \vee H^{n-2} \tag{6}$$

Thus A_2 gives the fifth and the sixth term of the left-hand side of (4), so that the proof is complete. ∎

We remark that the equations (4) are homogeneous so that they have the trivial solution $h^n = 0$, which is physically irrelevant. Likewise (1) has the trivial solution $f = 0$. Hence our problem is to find nontrivial solutions.

3. SOLUTIONS OF THE HERMITE EQUATIONS OF TRANSFER

3.1. The Recursive Solution Scheme

In this section we study the equations of transfer without regard to the Fokker–Planck equation. We recall briefly the method developed in Ref. 1. Equation (4) can be written in the following form:

$$\nabla \cdot h^{n+1} = -g^n, \qquad n \in \mathbb{N}_0 \tag{7,n}$$

with

$$g^n = \partial_t h^n + n\Pi_n(\eta^T - K_2) \cdot h^n + n(\nabla - K_1) \vee h^{n-1} - n(n-1)K_2 \vee h^{n-2} \tag{7',n}$$

From (7) one concludes immediately that the hierarchy of equations of transfer has to be solved recursively if it allows nontrivial solutions at all. Second, if the equations (7) have a solution each h^n can be written in the form

$$h^n = -Z^n g^{n-1} + w^n \tag{8}$$

with $\nabla \cdot w^n = 0$ where Z^n is an inverse operator of the divergence operator $\nabla \cdot$. More precise, Z^n is any linear operator such that the relations

$$\nabla \cdot Z^n \sigma = \sigma$$

and

$$Z^n \nabla \cdot \phi = \phi + \psi, \qquad \nabla \cdot \psi = 0 \tag{9}$$

hold for any tensor σ of degree $n-1$ in the range of the divergence operator $\nabla \cdot$ and for any continuously differentiable tensor ϕ of degree n.

For the existence of Z^n see Ref. 1. In the one-dimensional case one possible form of Z^n can be written down easily because all "tensors" are scalars. Let $a_n \in \tilde{\Omega}_t$ for all $t \in I$ and let $x \in \tilde{\Omega}_t$; then Z^n defined by

$$(Z^n\sigma)(x,t) = \int_{a_n}^{x} \sigma(y,t)\, dy \tag{10}$$

for continuous σ is obviously an inverse operator of ∂_x.

Now assume that a nontrivial recursive solution of (7) exists. Then we see from (8) and (7′,0) that h^1 is determined by w^1 and h^0. Likewise h^2 is determined by w^2, w^1, and h^0, and so forth. The result of these considerations is that there are operators Q^n such that each solution $(h^0, h^1, h^2, \ldots)$ of (7) can be written in the form

$$h^n = Q^n(w^0, \ldots, w^{n-1}) + w^n, \qquad n \in \mathbb{N} \tag{11}$$

with $w^0 = h^0$ and with $\nabla \cdot w^n = 0$, $n \in \mathbb{N}$. The operators Q^n are recursively defined by the described procedure.

In the foregoing considerations we have met two characteristic problems:

1. The problem of existence, i.e., under what conditions is a nontrivial recursive solution of the hierarchy (7) possible at all?
2. If a solution of (7) exists, which is the explicit form of the operators Q^n?

3.2. An Existence Theorem for the Equations of Transfer

In Ref. 1 we proved the following existence theorem for the equations of transfer.

Theorem. Let $\Omega = \tilde{\Omega} \times I$ be a compact set and let $\partial\Omega$ be a C^∞-Liapunov surface. Moreover let $K_1 \in C^\infty(\Omega)$, $K_2 \in C^\infty(\Omega)$ and $\omega^k \in C^\infty(\Omega)$ for all $k \in \mathbb{N}_0$. Then the hierarchy (7) can be solved recursively.

In this paper we will generalize this theorem in several respects, but on the other hand we will restrict our attention to one-dimensional systems. Such systems play a prominent role in many applications (Refs. 3, 4).

For the space-time region Ω we assume the following hypotheses to be valid:

1. $\Omega = \bigcup_{t \in I} \tilde{\Omega}_t \times \{t\}$ with $\tilde{\Omega}_t = [\alpha_t, \beta_t]$;
2. Ω is closed;
3. There is a number a such that $a \in \tilde{\Omega}_t$ for all $t \in I$.

It is useful to introduce the following class $B(\Omega)$ of functions.

Definition. $B(\Omega)$ is the set of all functions $g: \Omega \to \mathbb{R}$ with $\partial_t^r g \in C(\Omega)$ and $\partial_t^r \partial_x g \in C(\Omega)$, $r \in \mathbb{N}_0$.

Then the existence theorem reads as follows.

Theorem. Assume that $\partial_t^r K_\mu \in C(\Omega)$, $\mu \in \{1, 2\}$, $r \in \mathbb{N}_0$, and let $w^k \in B(\omega)$, $k \in \mathbb{N}_0$. Then, if g^n is defined by $(7', n)$ the equations

$$h^{n+1}(x,t) = -\int_{a_{n+1}}^{x} g^n(y,t)\,dy + w^{n+1}(x,t) \tag{12}$$

with $a_{n+1} \in \tilde{\Omega}_t$ for all $t \in I$ and $n \in \mathbb{N}_0$ can be solved recursively. Equation (12) yields a solution $(h^0, h^1, \ldots)$ of the equations of transfer (7) if w^k is independent of x, i.e., $w^k \in C^\infty(I)$, $k \in \mathbb{N}$.

Proof. Equation (12) for $n = 0$ reads

$$h^1(x,t) = -\int_{a_1}^{x} \partial_t w^0(y,t)\,dy + w^1(x,t)$$

Hence $\partial_t^r \partial_x h^1 \in C(\Omega)$. Moreover it is easily seen that

$$\partial_t^r h^1(x,t) = -\int_{a_1}^{x} \partial_t^{r+1} w^0(y,t)\,dy + \partial_t^r w^1(x,t)$$

Then by the hypotheses on Ω one finds $\partial_t^r h^1 \in C(\Omega)$, so that $h^1 \in B(\Omega)$.

Now assume that for $l \leqslant n$ we have $h^l \in B(\Omega)$. Then $\partial_t^k g^n \in C(\Omega)$ by definition of g^n, and we find from (12) for $n + 1$:

$$\partial_t^r \partial_x h^{n+1} = -\partial_t^r g^n + \partial_t^r \partial_x w^{n+1} \in C(\Omega)$$

and

$$\partial_t^r h^{n+1} = -\int_{a_{n+1}}^{\cdot} \partial_t^{r+1} g^n\,dy + \partial_t^r w^{n+1} \in C(\Omega)$$

Thus the functions h^n are defined and are in $B(\Omega)$ for all $n \in \mathbb{N}_0$. By construction they satisfy the equations $\partial_x h^{n+1} = -g^n + \partial_x w^{n+1}$. Hence the proof is complete. ■

Now the theorem can be stated another way using the operators Z^n defined by (10) and the operators Q^n defined by (11).

Corollary. Let Z^n be defined on $C(\Omega)$ and let $w^n \in B(\Omega)$. Then $(w^0, \ldots, w^{n-1})$ are in the domain of Q^n.

The proof follows immediately from that of the above theorem. In order to justify certain truncation procedures of the hierarchy (7) as we will do in Section 4 one has to start the recursive solution with $n = m + 1 > 0$ rather than with $n = 0$. From the proof of the above theorem one concludes immediately as follows.

Corollary. The Equations $(7, n)$ for $n > m + 1$ can be solved recursively if the functions $h^{m+1}, h^m, h^{m-1}, w^n$, $n > m + 1$ are in $B(\Omega)$.

Remark. Clearly one is not only interested in general solutions of (7) rather than in solutions satisfying certain initial-value and boundary-value conditions. We do not touch these problems here. But it is easily seen from (12) that h^n can be chosen such that $h^n(a_n, \cdot) = w^n$ is a given $C^\infty(I)$-function.

3.3. The Explicit Form of the Recursive Solutions

The operators Q^n are defined by the recursive procedure in a complicated way, and the existence theorems in the one-dimensional as well as in the three-dimensional case determine only subsets of the domains of the Q^n. Therefore it is clear that one can hardly give an explicit form of Q^n.

But we will give the explicit form of certain restrictions of them. We start with some notation.

Notation. Let Z^n, $n \in \mathbb{N}$ be operators as introduced in 3.1. From the existence theory of 3.2 we know that the domain of Z^n contains the $C^\infty(\Omega)$-tensors of degree $n-1$ or in the one-dimensional case the $B(\Omega)$-functions. Now we define the operators

$$\begin{aligned} S_n^{n+1} &= -Z^{n+1}\left[\partial_t \mathbf{1}\cdot + n\Pi_n\left(\eta^T - K_2\right)\cdot\right] \\ S_{n-1}^{n+1} &= -nZ^{n-1}(\nabla - K_1)\vee \\ S_{n-2}^{n+1} &= n(n-1)Z^{n-1}K_2 \vee \end{aligned} \tag{13}$$

where $\mathbf{1} = \sum_i e_i \otimes e_i$ means the unit tensor. Here the domains of the differential operators are understood to be $C^1(\Omega^0)$. The operator S_α^{n+1} acts on tensors of degree α. By means of (13) another set of linear operators is defined:

$$\begin{aligned} &q_n^n = \mathbf{1}, \qquad q_k^n = 0, \qquad n<k \\ &q_k^n = \sum_{(\alpha_1,\ldots,\alpha_r)\in\Lambda_k^n} S_{n-\alpha_1}^n S_{n-\alpha_1-\alpha_2}^{n-\alpha_1} S_{n-\alpha_1-\cdots-\alpha_r}^{n-\alpha_1-\cdots-\alpha_{r-1}} \end{aligned} \tag{14}$$

for $n>k$, $k\in\mathbb{N}_0$ where Λ_k^n is the set of all r-tuples $(\alpha_1,\ldots,\alpha_r)$ with $\alpha_j \in \{1,2,3\}$ and $\sum_{j=1}^r \alpha_j = n-k$.

With this notation the following representation theorem can be formulated.

Theorem. Q^n is given by

$$Q^n(w^0,\ldots,w^{n-1}) = \sum_{k=0}^{n-1} q_k^n(w^k) \tag{15}$$

whenever the right-hand side of (15) makes sense. This implies that

$$h^n = \sum_{k=0}^{n} q_k^n(w^k), \qquad n\in\mathbb{N}_0 \tag{15'}$$

is a solution of (7) whenever the right-hand side of (15′) is defined and $\nabla\cdot w^k = 0$, $k\in\mathbb{N}$.

Proof. First of all we note that the operators Q^k are recursively defined by

$$\begin{aligned} &Q^{n+1}(w^0,\ldots,w^n) \\ &\quad = -Z^{n+1}\Big\{\left[\partial_t + n\Pi_n\left(\eta^T - K_2\right)\cdot\right]\left[Q^n(w^0,\ldots) + w^n\right] \\ &\qquad + n(\nabla - K_1)\vee\left[Q^{n-1}(w^0,\ldots) + w^{n-1}\right] \\ &\qquad - n(n-1)K_2\vee\left[Q^{n-2}(w^0,\ldots) + w^{n-2}\right]\Big\} \end{aligned} \tag{16}$$

for all $(w^0, \ldots, w^n)$ for which the right-hand side is defined. Equation (16) can be written in the form

$$\begin{aligned} Q^{n+1}(w^0, \ldots, w^n) = S_n^{n+1}\big[\, Q^n(w^0, \ldots, w^{n-1}) + w^n\big] \\ + S_{n-1}^{n+1}\big[\, Q^{n-1}(w^0, \ldots, w^{n-2}) + w^{n-1}\big] \\ + S_{n-2}^{n-1}\big[\, Q^{n-2}(w^0, \ldots, w^{n-3}) + w^{n-2}\big] \end{aligned} \tag{16'}$$

whenever the right-hand side makes sense. By induction one proves that the q_k^n satisfy the equation

$$q_k^{n+1}(w^k) = S_n^{n+1} q_k^n(w^k) + S_{n-1}^{n+1} q_k^{n-1}(w^k) + S_n^{n+1} q_k^{n-2}(w^k) \tag{17}$$

From (17) one concludes that $\bar{Q}^n = \sum_{k=1}^{n-1} q_k^n$ satisfies also (16′). Since Q^n is the maximal operator defined by (16) $\bar{Q}^n$ is a restriction of Q^n. Thus the theorem holds. ■

4. THE TRUNCATED HIERARCHY

4.1. Justification of the Truncation Procedure

In thermomechanics one deals with the same systems as in kinetics with finitely many equations of transfer. This means that the whole thermomechanical information about a system is contained in the hierarchy of equations of transfer. The question is if it is possible to derive from the hierarchy a set of equations which contains only finitely many (Hermite) moments. Normally the "derivations" read as follows. Take $n > 0$ and put $h^k = 0$ for all $k > n$. Then the first $n + 1$ equations of transfer are assumed to be the thermomechanical field equations (Refs. 2 and 3). The solutions gained by this truncation procedure are regarded to be "approximations" of the exact solutions.

Now, for the case of the Fokker–Planck equation and its equations of transfer we want to show that the described truncation procedure can be justified rigorously. For this purpose let us consider the equations $(7, n)$ for $n = 0$ up to $n = m + 1$ and let $\nabla \cdot h^{m+1} = 0$ so that the mth equation simply reads

$$\begin{aligned} \partial_t h^m + m\Pi_m(\eta - K_2) \cdot h^m + m(\nabla - K_1) \vee h^{m-1} \\ - m(m-1)K_2 \vee h^{m-2} = 0 \end{aligned}$$

Hence we have $m + 1$ linear partial differential equations for the unknowns $(h^0, h^1, \ldots, h^m)$. Let us denote this system of equations by S_m. Then in the one-dimensional case one can state the following theorem.

Theorem. Let Ω, K_1, and K_2 satisfy the assumptions of the second theorem of Section 3.2 and let $(h^0, \ldots, h^m)$ be a solution of S_m with

$h^m \in B(\Omega)$ and $h^{m-1} \in B(\Omega)$. Then there are functions h^l, $l > m+1$ such that $(h^0, h^1, \ldots h^m, h^{m+1}, \ldots)$ is a solution of the complete hierarchy (7).

Proof. Let $h^{m+1}(x,t) = w^{m+1}(t)$ with $w^{m+1} \in C^\infty(I)$. Then $\partial_x h^{m+1} = 0$ and $h^{m+1} \in B(\Omega)$. Hence by the second corollary of Section 3.2 there are functions h^k, $k > m+2$ so that the equations $(7,k)$, $k > m+1$ are satisfied. The system S_m together with $\partial_x h^{m+1} = 0$ is identical with $(7,n)$ for $n \in \{0, \ldots, m\}$. Hence the theorem holds. ■

Likewise we have in the three-dimensional case the following theorem.

Theorem. Let Ω, K_1, and K_2 satisfy the hypotheses of the first theorem of Section 3.2 and let $(h^0, \ldots, h^m)$ be a solution of S_m with $h^m \in C^\infty(\Omega)$ and $h^{m-1} \in C^\infty(\Omega)$. Then there are tensors h^l, $l > m+1$ such that $(h^0, \ldots, h^m, h^{m+1}, \ldots)$ is a solution of $(7,n)$, $n \in \mathbb{N}_0$.

The proof is similar to that in the one-dimensional case.

From these theorems one sees immediately that the truncation procedure has nothing to do with an approximation of any kind. But it is to be emphasized that the exact solutions of the hierarchy (7) do not lead automatically to solutions of the Fokker–Planck equation. Rather one has to show the convergence of a certain series and some other properties of it (Ref. 1).

4.2. The Truncated Hierarchy

In Section 4.1 we saw that the truncation procedure leads to a system S_m of linear partial differential equations. Let us now consider this system in the one-dimensional case in more detail.

First we note that S_m can be written in matrix form. For this purpose let us introduce the column vector h by $h = (h^0, \ldots, h^m)^T$, and let the matrices B and C be defined by

$$B = \begin{pmatrix} 0 & 1 & & & & & \\ 1 & 0 & 1 & & & & \\ & 2 & 0 & 1 & & 0 & \\ & & \ddots & \ddots & \ddots & & \\ & 0 & & \ddots & \ddots & \ddots & \\ & & & & m-1 & 0 & 1 \\ & & & & & m & 0 \end{pmatrix} \tag{18}$$

and

$$C = \begin{bmatrix} 0 & 0 & & & & & \\ K_1 & K_2 - n & & & & & \\ 2K_2 & 2K_1 & 2(K_2-\eta) & & 0 & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & 0 & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & \ddots \\ & & & & m(m-1)K_2 & mK_1 & m(K_2-\eta) \end{bmatrix} \tag{19}$$

Then the system S_m simply reads

$$\partial_t h + B\partial_x h = Ch \tag{20}$$

It is classified by the following lemma.

Lemma. B has $m+1$ distinct real eigenvalues. Thus S_m is hyperbolic. For odd m all eigenvalues are nonzero, for even m one eigenvalue is zero.

Proof. Let $x = (x_0, \ldots, x_m)$ be an eigenvector of the matrix B to the eigenvalue λ. The components of x satisfy

$$x_{k+1} = \lambda x_k - kx_{k-1} \tag{21}$$

with $x_{m+1} = 0$ and $x_{-1} = 0$.

Since any eigenvector is only determined up to a constant factor, we may set $x_0 := 1$. Obviously by (21) for all $k \in \mathbb{N}_0$ numbers x_k are defined. Moreover x_k is a polynomial $\bar{x}_k(\lambda)$ in λ of degree k, and $x = (\bar{x}_0(\lambda_0), \ldots, \bar{x}_m(\lambda_0))$, is an eigenvector of B to the eigenvalue λ_0 if $\bar{x}_{m+1}(\lambda_0) = 0$. In what follows we therefore investigate the polynomials $\bar{x}_k$ especially the zeros of $\bar{x}_{m+1}$. Two properties of $\bar{x}_k$ are easily proved:

$$\frac{d\bar{x}_k}{d\lambda} = k\bar{x}_{k-1} \tag{22}$$

and

$$\text{if} \quad \bar{x}_k(\lambda) = 0 \qquad \text{then} \quad \bar{x}_{k-1}(\lambda) \neq 0, \qquad k \in \mathbb{N} \tag{23}$$

The first equation follows by induction differentiating (21).

The proof of the second equation runs as follows: The assertion is obviously true for $k = 1$. Let us assume that it holds up to $k = n$. If λ is a zero of $\bar{x}_{n+1}$ then by (21)

$$0 = \bar{x}_{n+1}(\lambda) = \lambda\bar{x}_n(\lambda) + n\bar{x}_{n-1}(\lambda)$$

Hence $\bar{x}_n(\lambda)$ must be unequal zero, for otherwise $\bar{x}_n$ and $\bar{x}_{n-1}$ would share the same zero in contradiction to the assumption.

From (22) and (23) it follows that $\bar{x}_k$ has k distinct real zeros. The proof runs again by induction. $\bar{x}_0 = 1$ has no zero and $\bar{x}_1 = \lambda$ the only zero $\lambda = 0$. Let $\lambda_1, \ldots, \lambda_k$ be the k distinct real zeros of x_k. Hence

$$\bar{x}_{k+1}(\lambda_i) = -k\bar{x}_{k-1}(\lambda_i) \neq 0 \tag{24}$$

and

$$\bar{x}_{k-1}(\lambda_i) = \frac{1}{k}\frac{d\bar{x}_k}{d\lambda}(\lambda_i) \neq 0 \tag{25}$$

Since the slopes at two adjacent zeros of a polynomial always differ in sign by (25) sign $\bar{x}_{k-1}(\lambda_i)$ and therefore by (24) sign $\bar{x}_{k+1}(\lambda_i)$ alternates from i to $i + 1$. Because of (22) the zeros of $\bar{x}_k$ are extrema of $\bar{x}_{k+1}$. This means that $\bar{x}_{k+1}$, which is a polynomial of degree $k + 1$, has k distinct extrema with positive values at its maxima and negative at its minima. From this we conclude that $\bar{x}_{k+1}$ has $k + 1$ distinct real zeros.

The eigenvalues of B are the roots of $\bar{x}_{m+1}$. Therefore B has $m + 1$ distinct real eigenvalues. ■

This lemma guarantees that one can apply the existence theory for hyperbolic systems. Especially in the stationary case we have explicit ordinary linear differential equations. From the very extended theory of these equations we extract the following results relevant to us.

Proposition. If the coefficients of C are in $C^{\infty}(\Omega)$ and if the initial conditions and the boundary conditions are also in $C^{\infty}(\Omega)$ then a solution h of S_m exists in Ω and is in $C^{\infty}(\Omega)$ (cf. Ref. 5).

In the stationary case the situation is much easier.

Proposition. Let $\tilde{\Omega}$ be independent of t and let the coefficients of C be in $C(\tilde{\Omega})$. Moreover let m be odd. Then for any given $r = (r_0, \ldots, r_m)$ and $x_0 \in \tilde{\Omega}$ a stationary solution h of S_m exists in $\tilde{\Omega}$ with $h(x_0) = r$ (e.g., Ref. 6).

For even m one eigenvalue of B is zero so that one has m differential equations and one algebraic equation. But with some care similar results can be proved. Combining these propositions with the theorems of Section 4.1 we arrive at the following result. For suitable initial conditions and boundary conditions the truncation procedure leads to (exact) solutions of the complete hierarchy of equations of transfer.

4.3. Solutions of the Truncated Hierarchy

In this section we assume that there are nontrivial solutions of S_m (in the one-dimensional case). We want to study the formal structure of these solutions.

The $(m+1)$th equation of S_m reads

$$\left[\partial_t - m(K_2 - \eta)\right]h^m = m(K_1 - \partial_x)h^{m-1} + m(m-1)K_2 h^{m-2} \tag{26}$$

Let us abbreviate the right-hand side of (26) by G^m. Since by assumption G^m belongs to the range of the operator $F^m = \partial_t - m(K_2 - \eta)$ we can write (26) in the form

$$F^m h^m = G^m \tag{27}$$

or equivalently by

$$h^m = L^m G^m + \psi^m \tag{28}$$

where L^m is an inverse of F^m and ψ^m is in the null space $N(F^m)$. Inserting (28) into the mth equation of S_m we arrive at

$$\partial_x L^m G^m + \partial_x \psi^m = g^{m-1} \tag{29}$$

Writing all terms containing h^{m-1} on the left-hand side defines an operator F^{m-1} and a function G^{m-1} such that (29) becomes

$$F^{m-1}h^{m-1} = G^{m-1} \tag{30}$$

where G^{m-1} depends on h^{m-2}, h^{m-3}, and ψ^m. If L^{m-1} denotes an inverse operator of F^{m-1}, (30) can be written in the form

$$h^{m-1} = L^{m-1}G^{m-1} + \psi^{m-1}$$

where ψ^{m-1} is an element of the null space $N(F^{m-1})$. This procedure can be carried on ending up with an equation

$$F^0 h^0 = G^0 \tag{31}$$

where G^0 depends on the elements $\psi^m, \psi^{m-1}, \ldots, \psi^1$ from the respective null spaces $N(F^m), \ldots, N(F^1)$.

Let us illustrate the described solution scheme for $m = 1$ where it is well known. The equation for h^1 reads

$$\partial_t h^1 + (\eta - K_2)h^1 = (K_1 - \partial_x)h^0 \tag{32}$$

Let us briefly write

$$\tilde{K}(x,t) = \int_{t_0}^{t} \left[\eta(x,t') - K_2(x,t') \right] dt'$$

Then the solution of (32) is of the form

$$h^1(x,t) = e^{-\tilde{K}(x,t)} h^1(x,t_0) + \int_{t_0}^{t} e^{[\tilde{K}(x,t') - \tilde{K}(x,t)]} \left[K_1(x,t') - \partial_x \right] h^0(x,t')\, dt' \tag{33}$$

Inserting (33) into the first equation of S_1 and denoting the particle density by $\rho = h^1$ we arrive at a generalized diffusion equation

$$\partial_t \rho(x,t) = \partial_x \int_{t_0}^{t} e^{\tilde{K}(x,t') - \tilde{K}(x,t)} \left[\partial_x - K_1(x,t') \right] \rho(x,t')\, dt' - \partial_x \left[e^{\tilde{K}(x,t)} h^1(x,t_0) \right] \tag{34}$$

This equation describes a diffusion process with memory and this again means that on the stage of the equations of transfer memory effects are possible. As far as we are aware such effects are not yet observed experimentally in diffusing systems.

5. FINAL REMARKS

In the Sections 2–4 we treated the first and the second problem stated in Section 1. Though we concede that some of the results could perhaps be sharpened, the solution of the two problems is almost satisfactory. Now, what can be said about the third problem? In Ref. 1 we treated this question for the so-called diffusion solutions. But mostly in physical literature the problem is not even mentioned. In this paper we do not try to give an answer; rather we confine ourselves to the remark that there are solutions of the equations of transfer which do not lead to solutions of the Fokker–Planck equation. This means that the truncation procedure says almost nothing about the Fokker–Planck equation itself. The third problem is a very difficult one which needs a special treatment.

APPENDIX

A.1. Symmetric Tensors

The action of the symmetrizer Π_n on a tensor of degree n is defined by its action on a generator $t_1 \otimes \cdots \otimes t_n$

$$\Pi_n t_1 \otimes \cdots \otimes t_n := \frac{1}{n!} \sum_{\sigma} t_{\sigma i_1} \otimes \cdots \otimes t_{\sigma i_n} \tag{35}$$

where σ runs over all permutations of $(1, \ldots, n)$. We also use the notation

$$t_1 \vee \cdots \vee t_n := \Pi_n t_1 \otimes \cdots \otimes t_n \tag{36}$$

A.2. The Inner Product

The inner product of a vector u and a tensor $T = t_1 \otimes \cdots \otimes t_n$ of degree n is defined by

$$u \cdot T := (u \cdot t) t_2 \otimes \cdots \otimes t_n$$

where $u \cdot t_1$ denotes the inner product of vectors. This notation can be easily generalized to an inner product of tensors of arbitrary degree

$$\begin{aligned}(u_1 \otimes \cdots \otimes u_m) \cdot (t_1 \otimes \cdots \otimes t_n)& \\ = (u_m \cdot t_1) u_1 \otimes \cdots \otimes u_{m-1} \otimes t_2 \otimes \cdots \otimes t_n&\end{aligned} \tag{37}$$

A.3. The ∇-Operator

The effect of ∇ on a function f or a tensor field T is given by

$$\begin{aligned}\nabla f &:= \sum_i \partial_i f e_i \\ \nabla T &:= \sum_i e_i \otimes \partial_i T\end{aligned} \tag{38}$$

∇^2 is defined iteratively

$$\nabla^2 T := \nabla(\nabla T) \tag{39}$$

Definition of the operator $\nabla \cdot$: The action of $\nabla \cdot$ on a tensor field T of degree $n > 1$ is given by

$$\nabla \cdot T := \sum e_i \cdot \partial_i T \tag{40}$$

Definition of the operator $\nabla \vee$: The action of $\nabla \vee$ on a tensor field T of degree n is given by

$$\nabla \vee T := \Pi_n(\nabla T) \tag{41}$$

The operator $K - \nabla$: Let K be a vector field; then the action of $K - \nabla$ on a tensor field T is defined by

$$\begin{aligned}(K - \nabla) T &:= K \otimes T - \nabla T \\ (K - \nabla) \cdot T &:= K \cdot T - \nabla \cdot T\end{aligned} \tag{42}$$

and

$$(K - \nabla) \vee T := K \vee T - \nabla \vee T$$

A.4. The Tensorial Hermite Functions and Hermite Polynomials

The nth tensorial Hermite function is defined by

$$\phi^n = -\nabla\phi^{n-1} \tag{43}$$

or

$$\phi^n = (-1)^n\nabla^n\phi^0$$

where ϕ^0 is given by

$$\phi^0(u) = (2\pi)^{-N/2}\exp\left(-\tfrac{1}{2}\|u\|^2\right) \tag{44}$$

N is the dimension of the vector space to which u belongs.

The Hermite functions satisfy the differential equation

$$\nabla\cdot(\nabla + u)\phi^n(u) = -n\phi^n(u) \tag{45}$$

and the relations

$$(\partial_i + u_i)\phi^n = ne_i \vee \phi^{n-1} \tag{46}$$

and

$$u\otimes\phi^n(u) = \phi^{n+1}(u) + n\sum_i e_i\otimes\left[e_i \vee \phi^{n-1}(u)\right] \tag{47}$$

From this it follows easily for a constant tensor η of degree 2

$$\nabla\cdot\eta\cdot(\nabla + u)\phi^n = -n\Pi_n\eta^T\cdot\phi^n \tag{48}$$

where T means the transposed of a tensor of degree 2.

Proof.

$$\begin{aligned}\nabla\cdot\eta\cdot(\nabla + u)\phi^n &= \sum_{i,j}\eta_{ij}\partial_i(\partial_j + u_j)\phi^n\\ &= n\sum_{i,j}\eta_{ij}e_j\vee\partial_i\phi^{n-1} = -n\sum_{i,j}\eta_{ij}e_j\vee(e_i\cdot\phi^n)\\ &= -n\Pi_n(\eta^T\cdot\phi^n)\end{aligned}$$

The connection between Hermite functions and Hermite polynomials is established by

$$H^n := \phi^n/\phi^0 \tag{49}$$

Some useful relations for Hermite polynomials are listed below:

$$H^{n+1}(u) = (u - \nabla)H^n(u) \tag{50a}$$

$$H^n(u) = (u - \nabla)^n 1 \tag{50b}$$

$$\partial_i H^n = ne_i \vee H^{n-1} \tag{50c}$$

$$(u\cdot\nabla - \nabla\cdot\nabla)H^n(u) = nH^n(u) \tag{50d}$$

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